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Improved Predictor Corrector Scheme for Solving Cox-Inegrsoll-Ross Interest Rate Model: A Comprehensive Analysis and Applications in Financial Modeling

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Abstract

An Uncertain Differential Equation (UDE) is a type of differential equation driven by Liu's canonical process. It has always been a tough problem to obtain the analytic solution of UDE. In this paper, we study a new numerical method for solving UDEs: the Improved Predictor-Corrector (IPC). Also, we translate a UDE into a system of ODEs using the concept of an α-path, which is a certain type of function. Moreover, the convergence and stability of the IPC method are detailed. This method has many applications in Financial mathematics from a numerical perspective. Furthermore, this research comprehensively analyzes the Cox-Ingersoll-Ross (CIR) interest rate model and explores its application in financial modeling. The CIR model is widely used in finance to model interest rates and has proven to be a valuable tool for understanding and predicting interest rate dynamics. Through comprehensive analysis and exploration of the interest rate model, the CIR aims to deepen understanding of interest rates, examine its application across various financial contexts, and provide deeper insight into the crucial role of interest rates in effective financial decision-making. Finally, we present various examples to show our assertions.

Keywords: Fuzzy canonical Liu's process, Improved predictor corrector method, Cox Ingersoll-Ross model.

1 | Introduction

Most phenomena and events in the real world occur unexpectedly, among them changes in economic and political systems, the collapse of governments, conflicts between tribes, wars, and terrorist attacks. Thus, it is not possible to accurately anticipate or estimate the prices of stocks, valuable papers, monetary units, and precious metals. Therefore, the only way to find out how this factor can affect the decline in the value of the

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company is to focus on stock prices. Investigation into the effects of the factors, along with uncertainty theory, can help provide a better understanding and more precise modeling of these phenomena. The uncertainty theory was first introduced by Liu [1], who then presented the concept of an uncertainty measure, which is a powerful tool for dealing with uncertain phenomena and facilitates the measurement of events based on normality, monotonicity, self-duality, and maximality axioms. Then the concept of an uncertain process was proposed by Liu [2], introducing a particular uncertain process with stationary, dependent increments, named the canonical Liu process, which is similar to a stochastic process described by Brownian motion. Since then, some literature has been published on Liu's canonical process and its applications in other sciences, such as economics and optimal control [3]. Then Liu [2] was inspired by stochastic notions and the Ito process to introduce Uncertain Differential Equation (UDE) driven by a canonical Liu process for a better understanding of uncertain phenomena.

Regarding the importance of existence and uniqueness of a solution to UDEs driven by canonical Liu's process, Chen and Liu [4] investigated the existence and uniqueness of a solution to the UDEs by employing Lipschitz and Linear growth conditions, and stability analysis of UDEs was given by Yao et al. [5]. Many researchers have found analytic solutions for certain types of UDEs, such as Chen and Qin [6]. From many perspectives, it is difficult to find analytic solutions to UDEs. Therefore, it is of great importance to develop numerical methods for solving UDEs. In the last few years, many works have been published by several authors on numerical solutions of UDEs.

As some examples to mention, Yao and Shen [7], [8] have investigated the numerical solution of UDEs using the Euler and Runge-Kutta methods, respectively. Also, the numerical solutions of UDEs using the Adams method [9] and the Adams-Simpson method have been studied. In this paper, we aim to develop a new numerical method for UDEs that is more accurate than those in [9] and [10]. First, we investigate the concept of an α-path to UDE, a type of function that solves the associated ordinary differential equation. Then, an Improved Predictor-Corrector (IPC) method is designed to solve UDEs, which essentially solves each α-path and produces an inverse uncertainty distribution for the solution. The IPC method is generated by combining an explicit three-step method and an implicit two-step method. The remainder of this paper is organized as follows: Section 2 introduces basic concepts and theorems in uncertainty theory and UDE. An explicit three-step method, an implicit two-step method for solving UDEs, and the IPC three-step algorithm are proposed. We introduce a brief survey of fundamental concepts from financial mathematics, and present some numerical examples in Section 5. The last section contains a summary.

2 | Preliminaries

In this section, we introduce some basic concepts and theorems of uncertainty theory and UDE driven by the canonical Liu process, which are used throughout this paper.

2.1 | Credibility Theory

Suppose that Θ is a nonempty set and P is the power set of Θ. Each element of A in P is called an event. To present an axiomatic definition of uncertainty, it is necessary to assign a number M{A} to each event A, which indicates the uncertainty that A will occur. To ensure that the number M{A} has certain mathematical properties which we intuitively expect to have an uncertainty, we accept the following four axioms [1].

I. Axiom (normality) $\mathrm{M}\{\Theta\} = 1$

II. Axiom (monotonicity) $\mathrm{M}\{A\} \leq \mathrm{M}\{B\}$ whenever A ⊂ B.

III. Axiom (self-duality)

IV. Axiom (maximality) $\mathrm{M}\{A\} + \mathrm{M}\{Ac\} = 1$ for any event A. $\mathrm{M}\{UiAi\} = \mathrm{supiM}\{Ai\}$ for any events {Ai} with supi $\mathrm{M}\{Ai\} < 0.5$.

Definition 1 ([1]). The set function M is called an uncertainty measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms. Afamily P with these four properties is called a σ-algebra. The pair (Θ, P) is called a measurable space, and the elements of P are afterwards called P-measurable sets instead of events.

Definition 2 ([1]). Let Θ be a nonempty set, P the power set of Θ, and M an uncertainty measure. The triple (Θ, P, M) is called an uncertainty space. Let (Θ, P, M) be an uncertainty space. A filtration is a family $\{P_t\}t \geq 0$ of increasing sub-σ algebras of P (i.e. $P_t \subset Ps \subset P$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $P_t = \cap_{s>t} P_s$ for all t ≤ 0. When the uncertainty space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and P_0 contains all M-null sets. We also define $P_\infty = \sigma(U_{t\geq 0} P_t)$ (i.e., σ-algebra generated by $U_{t\geq 0} P_t$.) A measurable uncertain variable is denoted by $Lp(\Theta, Rd)$, which will be defined later. A process is called P-adapted if, for all $t \in [0, t]$, the uncertain variable x(t) is P-measurable.

Definition 3 ([1]). An uncertain variable is defined as a (measurable) function $\xi: (\Theta, P, M) -\!\!\rightarrow R$.

Definition 4 ([1]). Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int +\infty\ 0\ M\{\xi \geq r\}dr - \int 0\ -\infty M\{\xi \leq r\}dr.$$

provided that at least one of the two integrals is finite. Furthermore, the variance is defined by $E[(\xi - e)2]$. Let ξ and η be independent uncertain variables with finite expected values. Then for any numbers a and b, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Definition 5 ([10]). The uncertainty distribution μ(x) of an uncertainty variable ξ is defined by

$$\mu(x) = \max\{1,2M(\xi = x)\}, x \in R.$$

Definition 6 ([10]). An uncertainty distribution μ(x) is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \mu(x) < 1$, and

$$\lim x \rightarrow -\infty \mu(x) = 0, \lim x \rightarrow +\infty \mu(x) = 1.$$

In addition, the inverse function $\mu-1(\alpha)$ is called the inverse uncertainty distribution of ξ. In the following expression, we use the notation x(t) instead of xt(θ). An uncertain process is essentially a sequence of uncertain variables indexed by time or space. As one of the most important types of uncertain processes, the canonical Liu process is defined as follows.

Definition 7 ([11]). An uncertain process C_t is said to be a canonical Liu's process if

I. $C_0 = 0$,
II. C_t has stationary and independent increments,
III. Every increment C_t+s-C_s is a normally distributed uncertain variable with expected value et and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = \left(1 + \exp\left(\pi|e - x|\sqrt{}3\sigma\right)\right) - 1, -\infty < x < +\infty.$$

The parameters e and σ are called the drift and diffusion coefficients, respectively. In addition, a canonical Liu process is said to be standard if e = 0 and σ = 1. Based on Liu's canonical process, Liu's integral is defined as an uncertain counterpart of the Ito integral as follows.

Definition 8 ([9]). Suppose that x(t) is an uncertain process and C_t is a canonical Liu's process. For any partition of the closed interval [a,b] with $a = t_1 < t_2 < \ldots < t_k + 1 = b$, the mesh is written as

$\Delta = \max 1 \leq i \leq k |ti + 1 \ - ti|$.

Then the Liu integral of x(t) with respect to C_t is a canonical Liu process. For any partition of the closed interval [a, b] with $a \ = t_1 < t_2 < \ldots < t_k + 1 = b$, the mesh is written as

$\Delta = \ \max 1 \leq i \leq k \, |t_i + 1 \ - t_i|$.

Then the Liu integral of x(t) with respect to C_t is

$$\int_a^b x(t) dC_t = \lim_{\Delta \to 0} \ \sum k_{i=1} x(t_i).(C_{ti+1} - C_{ti}).$$

provided that the limit exists almost surely and is an uncertain variable.

2.2 | Uncertain Differential Equation

In 2008, Liu [1] presented a type of UDEs driven by the canonical Liu's process.

Definition 9 ([1]). Suppose C_t is a canonical Liu's process and f, g are some given functions,

$$dx(t) = f\,(x(t), t) d_t + g(x(t), t) d_{Ct}. \quad \textbf{(1)}$$

is called a UDE with an initial value x(0). The solution is an uncertain process x(t) that Satisfies (2) identically in t. Gao [12] proved the existence and uniqueness theorem of the solution of UDEs under linear growth conditions and Lipschitz continuous conditions.

Theorem 1 ([6]). The UDE

$$dx(t) = f\,(x(t), t) d_t + g(x(t), t) d_{Ct}.$$

has a unique solution if the coefficients f (t, x) and g(t, x) satisfy the linear growth condition

$$|f\,(t, x)| + |g(t, x)| \ \leq \ L(1 + |x|) \text{for all } x \in R, \qquad t \geq 0.$$

And Lipschitz condition

$$|f\,(t, x) - f\,(t, y)| + |g(t, x) - g(t, y)| \leq L|x - y|, \text{for all } x, y \in R, \qquad t \geq 0.$$

for some constant L. Moreover, the solutionis sample continuous.

Definition 10 ([7]). The α-path (0 < α < 1) of an UDE

$$dx(t) = \ f\,(x(t), t) dt \ + \ g(x(t), t) dC_t.$$

with initial value x(0) is a deterministic function $x^\alpha(0)$ with respect to t that solves the corresponding ordinary differential equation

$$dx^\alpha(t) = f\,(x_\alpha(t), t) d_t + \ |g(x^\alpha(t), t)| \mu^{-1}(\alpha) d_t, \quad \textbf{(2)}$$

where

$$\mu^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln(\frac{\propto}{\propto -1}).$$

Theorem 2 ([7]). Let x(t) and $x^\alpha(t)$ be the solution and α-path of uncertain differential *Eq. (1)*, respectively. Then

$$M\{x(t) \leq x^{\alpha}(t), \text{for all } t\} = \alpha,$$

$$M\{x(t) > x^{\alpha}(t), \text{for all } t\} = 1 - \alpha.$$

3 | Improved Predictor Corrector Method for Uncertain Differential Equation

Theorem 2 can be used to numerically solve the UDE *Eq. (1)* by applying any suitable numerical method for ODEs to *Eq. (2)*. The IPC method is generated by combining an explicit three-step method and an implicit two-step method. This argument also improves on [8], [9] which are considered in the Adams and Adams-Simpson methods, respectively, for *Eq. (2)*. We first introduce the IPC method for ordinary differential equations and then apply it to UDEs.

3.1 | Improved Predictor Corrector Method

For an ordinary differential equation with initial value x(0)

$$dx(t) = f(t, x(t))d_t,$$

by linear spline interpolation for $f(t_i - 1, x(t_i - 1)), f(t_i, x(t_i)), f(t_i + 1, x(t_i + 1)$ The explicit three-step method is obtained as follows:

$$\begin{aligned} X_{(t_i+2)} = X_{(t_i-1)} &+ \frac{h}{2}\left[f\left(t_i - 1, x_{(t_i-1)}\right) + f\left(t_i, x(t_i)\right)\right. \\ &\left. + 4f\left(t_i + 1, x_{(t_i+1)}\right)\right], \end{aligned} \tag{3}$$

$$x(t_i - 1) = \alpha_0,\ x(t_i) = \alpha_1,\ x(t_i + 1) = \alpha_2.$$

Also, the implicit two-step method can be written as:

$$x(t_{i+1}) = x(t_{i-1}) + \frac{h}{2}[f(t_{i-1}, x(t_{i-1})) + 2f(t_i, x(t_i)) + f(t_{i+1}, x(t_{i+1}))], \tag{4}$$

$$x(t_i - 1) = \alpha_0, x(t_i) = \alpha_1.$$

Using the relation between the explicit three-step method as a predictor and the implicit two-step method as a corrector. Now, we consider the UDE.

$$dx(t) = f(x(t), t)d_t + g(x(t), t)d_{Ct}.$$

with initial value x(0). According to *Theorem 1*, the UDE *Eq. (1)* has a unique solution, and its α-path ($0 < \alpha < 1$) is

$$dx^{\alpha}(t) = f(x^{\alpha}(t), t)d_t + |g(x^{\alpha}(t), t)|\mu^{-1}(\alpha)d_t, \tag{5}$$

$$x^{\alpha}(0) = x(0).$$

For the sake of simplicity, we write

$$F(t, x^{\alpha}(t)) = f(t, x^{\alpha}(t)) + |g(t, x^{\alpha}(t))|\mu^{-1}(\alpha).$$

Then the ordinary differential *Eq. (5)* is reduced to

$$dx^{\alpha}(t) = F\left(t, x^{\alpha}(t)\right)d_t,$$

$x^{\alpha}(0 = x(0).$

We will present the IPC method for solving the uncertain differential *Eq. (1)* as follows. Algorithm (IPC three-step method for UDEs) To approximate the solution of the following initial value problem.

$$x'(t) = F\,(t, x(t)), t_0 \leq t \leq T,$$

$$x^{\alpha}(t_0) = \alpha_0, x^{\alpha}(t_1) = \alpha_1, x^{\alpha}(t_2) = \alpha_2$$

An arbitrary positive integer N is chosen.

Step 1. Let $h = \frac{T - t_0}{N}$,

$$W^{\alpha}(t_0) = \alpha_0, W^{\alpha}(t_1) = \alpha_1, W^{\alpha}(t_2) = \alpha_2$$

Step 2. Let i = 1.

Step 3. Let

$$\{W^{(0)\alpha}(t_i + 2) = W^{\alpha}(t_i - 1) + \frac{h}{2}\left[F^{\alpha}\big(t_i - 1, w(t_i - 1)\big) + F^{\alpha}\big(t_i, w(t_i)\big) + 4F^{\alpha}\big(t_i + 1, w(t_i + 1)\big)\right]$$

Step 4. Let ti + 2 = t_0 + (i + 2)h,

Step 5. Let

$$\{\, W^{\alpha}(t_i + 2) = W^{\alpha}(t_i) + \left(\frac{h}{2}\right)F^{\alpha}\big(t_i, w(t_i)\big) + hF^{\alpha}\big(t_i + 1, w(t_i + 1)\big) + \left(\frac{h}{2}\right)F^{\alpha}\left(t_i + 2, W^{(0)}(t_i + 2)\right)$$

Step 6. i = i + 1.

Step 7. If i ≤ N − 2, go to *Step 3*.

Step 8. The algorithm terminates, and wα(T) approximates the true value of X^{α}(T).

4 | Convergence

To integrate the system given in *Eq. (1)* from t_0 to a prefixed T > t_0, the interval [t_0, T] will be replaced by a set of discrete, equally spaced grid points $T_0 < T_1 < T_2 < \ldots < T_N = T$, and the exact solution X(t, α) is approximated by some x(t, α). The exact and approximate solutions at t_n, $0 \leq n \leq N$, are denoted by $X_n(\alpha)$ and $x_n(\alpha)$, respectively. The grid points at which the solution is calculated are $tn = t_0 + nh$, $h = (T - T_0)/N$, $1 \leq n \leq N$. From *Eq. (4)*, the polygon curves

$$x(t, h, \alpha) = \{[t_0, X_0(\alpha)], [t_1, X_1(\alpha)], \ldots, [t_N\,, X_N\,(\alpha)]\}$$

are the implicit three-step approximates to X(t, α) over the interval $t_0 \leq t \leq t_N$. The following lemma will be applied to show the convergence of these approximates, i.e.,

$$\lim_{h\to 0} x(t, h, \alpha) = X(t, \alpha)$$

Lemma 1. Let a sequence of numbers ${W_n}_{n=0}^{N}$ satisfy:

$$|w_{n+1}| \leq A|w_n| + B|w_{n-1}| + C, \quad 0 \leq n \leq N-1$$

For some given positive constants A, B, and C, then

$$w_n | \leq (A^{n-1} + \beta_1 A^{n-3}B + \beta_2 A^{n-5}B^2 + \ldots + \beta_s B[\tfrac{n}{2}])| w_1|+(A^{n-2}B + \gamma A^{n-4}B^2 + \ldots + \gamma AB\tfrac{n}{2})|W_0| + (A^{n-2} + A^{n-3} + \ldots + 1)C+ (\delta_1A^{n-4} + \delta_2A^{n-5} + \ldots + \delta_mA + 1)BC + (\zeta_1A^{n-6} + \zeta_2A^{n-7} + \ldots + \zeta_lA + 1)B^2C+ (\lambda_1A^{n-8} + \lambda_2A^{n-9} + \ldots + \lambda_pA + 1)B^3C + \ldots$$

When n is odd, and

$$w_n | \leq (A^{n-1} + \beta A^{n-3}B\beta_2 A^{n-5}B^2 + \ldots + \beta_s AB\tfrac{n}{2} - 1)|w | +(A^{n-2}B + \gamma A^{n-4}B^2 + \ldots + \gamma_t B\tfrac{n}{2})|W_0| + (A^{n-2} + A^{n-3} + \ldots + 1)C+ (\delta_1A^{n-4} + \delta_2A^{n-5} + \ldots + \delta_mA + 1)BC + (\zeta_1A^{n-6} + \zeta_2A^{n-7} + \ldots + \zeta_lA + 1)B^2C+ (\lambda_1A^{n-8} + \lambda_2A^{n-9} + \ldots + \lambda_pA + 1)B^3C + \ldots,$$

where β_s, γ_t, δ_m, ζ_l, and λ_p are constants for all s, t, m, l, and p.

Theorem 3. For any arbitrary fixed α: $0 \leq \alpha \leq 1$, the implicit two-step approximations of *Eq. (4)* converge to the exact solutions $X(t, \alpha)$ for $X \in C^3[t_0, T]$.

Proof: it is sufficient to show

$$\lim_{h\to 0} x_N(\alpha) = X(T, \alpha)$$

By using Taylor's theorem, we have:

$$X_{n+1}(\alpha) = X_n(\alpha) + \frac{h}{2} f(t_{n-1}, X_{n-1}(\alpha)) + hf(t_n, X_n(\alpha)) + \frac{h}{2} f(t_{n+1}, X_{n+1}(\alpha)) - \frac{1}{6} h^3X'''(\xi_n),$$

where $t_n < \xi_n$. Consequently

$$X_{n+1}(\alpha) - x_{n+1}(\alpha) = X_{n-1}(\alpha) - x_{n-1}(\alpha) + \frac{h}{2}\{f(t_{n-1}, X_{n-1}(\alpha)) - f(t_{n-1}, x_{n-1}(\alpha))\}+ h\{f(t_n, X_n(\alpha)) - f(t_n, x_n(\alpha))\} + \frac{h}{2}\{f(t_{n+1}, X_{n+1}(\alpha)) - f(t_{n+1}, x_{n+1}(\alpha))\} - \frac{1}{6} h^3X'''(\xi_n)$$

Denote $w_n = X_n(\alpha) - x_n(\alpha)$, Then

$$|w_n + 1| \leq hL_1|w_n| + (1 + \frac{hL_2}{2})|w_n - 1| + (\frac{hL_3}{3})|w_n + 1| + \frac{1}{6} h^3M,$$

where $M = \max_{t_0\leq t\leq T}|X'''(t, \alpha)|$. Set

$$L = \max\{L_1, L_2, L_3\} < \frac{1}{h},$$

Then

$$|w_n + 1| \leq \left(\frac{2hl}{1 - hl}\right)|w_n| + \left(\frac{1 + hl}{1 - hl}\right)|w_n - 1| + \left(\frac{1}{3 - 3hl}\right)h^3M$$

are obtained, if $h \to 0$ then $w_n \to 0$, which concludes the proof.

Theorem 4. For any arbitrary fixed α:$0 \leq \alpha \leq 1$, the explicit three-step approximations of *Eq. (3)* converge to the exact solution $X(t, \alpha)$ for $X \in C^3[t_0, T]$.

Proof: similar to *Theorem 4.*

Theorem 5. The explicit three-step method is stable.

Proof: for the explicit three-step method, there exists only one characteristic polynomial $p(\lambda) = \lambda^3 - \lambda$, then it satisfies the root condition and, therefore, it is a stable method.

Theorem 5. The implicit two-step method is stable.

Proof: similar to *Theorem 5*. Regarding the above-mentioned theorems, it is obvious that the IPC Three-step method is convergent and stable.

5 | Numerical Experiments

UDE have become standard models for financial quantities such as asset prices, stock prices, interest rates, and their derivatives. We will begin with a brief survey of the most fundamental concepts of financial mathematics.

Interest rate

A rate is charged or paid for the use of money. Also, an interest rate is often expressed as an annual percentage of the principal. It is calculated by dividing the amount of interest by the amount of principal. Interest rates often change in response to inflation and Federal Reserve policies.

Volatility

The relative rate at which the price of a security moves up and down. Volatility is calculated as the annualized standard deviation of daily price changes. If the price of a stock moves up and down rapidly over short time periods, it has high volatility. If the price changes seldom, it has low volatility.

Mean reversion

Mean reversion is the tendency for a variable to remain near its long-run average or to return to that average. For example, interest rates and implied volatilities tend to exhibit mean reversion. Exchange rate and stock prices tend not to. Stock market returns, however, do tend to exhibit mean reversion.

In the Cox, Ingersoll, and Ross (CIR) model, the short-term interest rate is given by:

$$dx(t) = (\theta - kx(t))dt + \sigma^* \sqrt{(x(t))} dC_t, \quad (6)$$

where we interpreted the above terms to mean

$dx(t)$: the change in the short-term interest rate.

k: the speed of mean reversion. It approximately equals 1-b, where b is the first-order autocorrelation.

The parameter σ^* no longer represents the volatility of interest rate changes. This argument is an example

of conditional volatility: when the short rate x(t) is high, the volatility of interest rate changes is high, and vice versa. According to for σ and σ^* we have $\sigma = \sigma^* \sqrt{\theta}$.

θ: the average interest rate.

σ: the volatility of the short rate.

d_{C_t}: a random term so that $E(d_{C_t}) = 0$. Expected changes in interest rates are given by

$$E(dx(t)) = (\theta - kx(t))d_t.$$

Example 1. Consider the following CIR for $\theta = 0.05, k = 0.2$, and $\sigma *= 0.08$, which represent reasonable values describing the year-to-year behavior of the interest rate. with given initial value $x(0) = 0$. The exact solution of *Eq. (5)* is

$$dx(t) = \left(0.05 - 0.2x(t)\right)d_t + 0.08\sqrt{x(t)dC_s}.$$

and its α-path is

$$dx^{\alpha}(t) = (\theta - kx^{\alpha}(t) + \sigma^{*}\mu^{-1}(\alpha))d_{t}.$$

Numerical solution of these equations is considered in *Fig. 1*, for step size=0.01, iteration number=1000, and T=50 years. From *Fig. 1*, we can see that the three uncertainty distributions produced respectively by Runge-Kutta, Adams, and IPC methods are very close to the uncertainty distribution of the analytic *Solutions (6)*. Hence, Adams, Runge-Kutta, and IPC methods can solve the UDE. We further observe in *Fig. 1* that the uncertainty distribution from the IPC method is closer to the analytical solution than that from the Adams method.

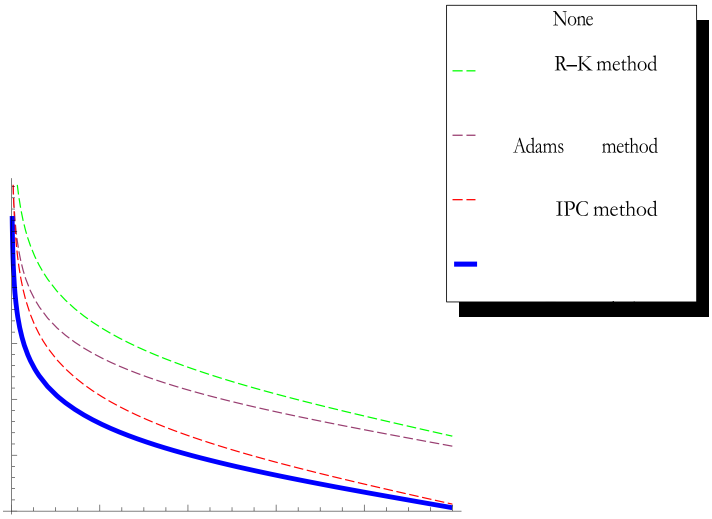

Fig. 1. The errors for r = 0.5 in Example 1.

and the Runge-Kutta method, respectively. Therefore, it is clear that the IPC method has higher accuracy than the Adams and Runge-Kutta methods. From the above analysis, we can see that the IPC method is a better approach for solving the UDE than the Adams and Runge-Kutta methods.

In *Table 1*, a comparison of the numerical results for the absolute errors obtained using the Runge-Kutta, Adams, and IPC methods with exact *Solution (6)* is given for the same step size h=0.01.

Table 1. Comparison of errors in the numerical solution using the Rung-Kutta, Adams, and IPC methods for Eq. (6) with iteration K = 1000 and step size h = 0.01.

α	Error (Rung-Kutta)	Error (Adams)	Error (IPM)
0.1	0.0938220508540224‘	0.060768275064677746‘	0.03242570697533298‘
0.2	0.08624662223500101‘	0.05930719157873898‘	0.025014159350041698‘
0.3	0.0815564403281559‘	0.058402589651273984‘	0.020425441229646846‘
0.4	0.07802522827778158‘	0.05772151985686069‘	0.016970620703632422‘
0.5	0.07510864235613368‘	0.05715899386133605‘	0.014117129957607832‘
0.6	0.0725607598371284‘	0.056667580205442825‘	0.011624366368516825‘
0.7	0.07024662199326381‘	0.05622124921113225‘	0.009360290802122417‘
0.8	0.0680813594558759‘	0.05580363200487204‘	0.007241869888378494‘
0.9	0.06600510737779208‘	0.05540318237007291‘	0.005210533850152244‘
1	0.06397066247726624‘	0.055010796143932894‘	0.0032201005651738557‘

Example 2. Consider the following CIR equations for $\theta = 0.0707$, $k = 0.6067$, and $\sigma * = 0.2928$.

$$dx(t) = (\theta - kx(t))dt + \sigma * \sqrt{(x(t))dCt.} \quad (7)$$

With the given initial value x(0) = 1. The explicit solution of *Eq. (7)* is

$$x(t) = x(0)\exp\big(at + bC(t)\big).$$

and its α-path is

$$dx^{\alpha}(t) = (\theta - kx^{\alpha}(t) + \sigma^{*}\mu^{-1}(\alpha))d_t.$$

The inverse uncertainty distribution of x(t) is We choose the parameters as follows, a = 0.5, b = 1.5, t = 10, N = 1000, $h = \frac{t}{N} = 0.01$.

The exact numerical solutions obtained by the Runga-Kutta method, the Adams method, and the present solution using IPM are tabulated in *Table 2* for h = 0.01. Also, the obtained errors are incorporated in this table. From the results, one may conclude that the solution obtained by the IPC method is identical to the exact solution. The solution plot is shown in *Fig. 2*. Using the proposed method in Section 3, we can obtain an approximate solution for this example. A comparison of the numerical results with the exact solution is shown in *Fig. 2*.

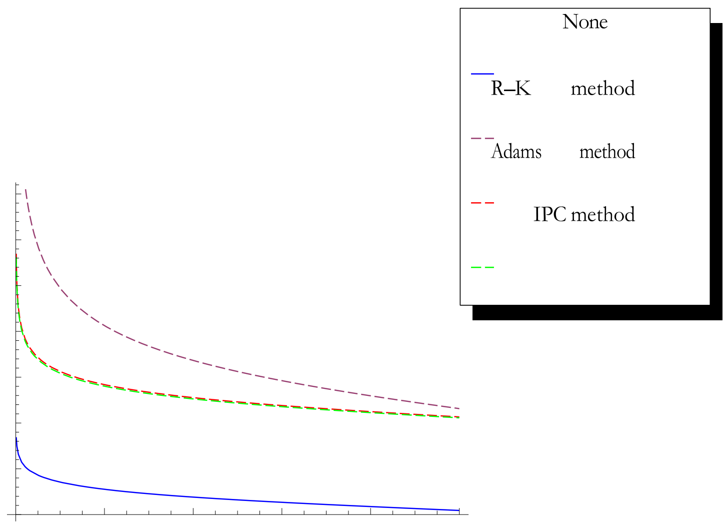

Fig. 2. The errors for *r* = 0.5 in Example 2.

Table 2. Comparison of errors in the numerical solution using the Rung-Kutta, Adams, and IPC methods for Eq. (6).

α	Error (Rung-Kutta)	Error (Adams)	Error (IPM)
0.1	1.808042193580918‘	0.9281769985563846‘	0.01954965844490064‘
0.2	1.587959722898813‘	0.6608581705328824‘	0.016895197550903962‘
0.3	1.4582751984957965‘	0.5178605149950599‘	0.015306752965624337‘
0.4	1.3637393146983499‘	0.4200838786965364‘	0.014136782818382265‘
0.5	1.2875751687724049‘	0.3451302959188256‘	0.013186493458830162‘
0.6	1.2224058081972575‘	0.28362043368009693‘	0.012367768392147704‘
0.7	1.1642836026596803‘	0.23074613350490614‘	0.011633093392665605‘
0.8	1.110794640360493‘	0.18369618391995957‘	0.010953162665008342‘
0.9	1.0602941039998126‘	0.14065458879409065‘	0.01030778902043017‘
1	1.0115406146754944‘	0.10034118393041669‘	0.00968151972950304‘

6 | Conclusion

UDE is an important tool for dealing with dynamic systems in an uncertain environment. In this paper, an IPC method is adapted and modified to solve UDEs. Numerical examples and comparisons with exact solutions show that the proposed algorithm produces accurate results.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All data are included in the text.

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