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Bayesian HyperNetwork and Bayesian SuperHyperNetwork using PowerSet and n th-Powerset

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Abstract

Graph theory offers a powerful framework for modeling relationships among entities; in dentistry, for example, it can represent connections between teeth or other oral structures. A *hypergraph* extends the classical graph by allowing *hyperedges* to join more than two vertices, thus capturing complex multiway interactions. A *superhypergraph* builds on this idea by introducing recursively nested powerset layers, enabling hierarchical and self-referential relationships among hyperedges. In parallel, *hypernetworks* and *superhypernetworks* generalize network models to these richer connectivity patterns. And a Bayesian network is a directed graph where nodes represent random variables and edges encode conditional dependencies via probability distributions.

In this work, we introduce the concepts of *Bayesian hypernetworks* and *Bayesian superhypernetworks*, which extend Bayesian networks by leveraging hypernetwork and superhypernetwork structures. These novel frameworks enhance the ability to model hierarchical and intricate real-world phenomena, offering significant advantages for complex decision-making and inference. We anticipate that their integration into Bayesian network theory and artificial intelligence will open new avenues for advanced probabilistic modeling and analysis.

Keywords: Bayesian hypernetwork, Bayesian Superhypernetwork, HyperGraph, Superhypergraph

1|Preliminaries

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. All concepts considered herein are assumed to be finite. Furthermore, unless otherwise specified, n is assumed to be a natural number.

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1.1|SuperHyperGraph

Graph theory offers a powerful framework for modeling relationships among entities; in dentistry, for example, it can represent connections between teeth or other oral structures [1, 2]. In classical graph theory, a hypergraph extends the idea of a conventional graph by permitting edges—called hyperedges—to join more than two vertices. This broader framework enables the modeling of more intricate relationships between elements, thereby enhancing its utility in various fields [3, 4, 5]. A *SuperHyperGraph* is an advanced extension of the hypergraph concept, integrating recursive powerset structures into the classical model. This concept has been recently introduced and extensively studied in the literature [6, 7, 8, 9, 10]. Including related concepts, we describe them below.

Definition 1 (Base Set). A *base set* S is the designated set from which subsequent constructions (e.g., powersets and hyperstructures) are formed. Formally,

$$S = \{x \mid x \text{ lies in the specified domain}\}.$$

All elements of constructions such as $\mathcal{P}(S)$ and $\mathcal{P}_n(S)$ ultimately originate from elements of S .

Definition 2 (Powerset). [11] The *powerset* of a set S , denoted $\mathcal{P}(S)$, is the set of all subsets of S :

$$\mathcal{P}(S) := \{A \mid A \subseteq S\}.$$

It contains both $\emptyset$ and S itself.

Definition 3 (n -th Powerset). (cf. [12, 13, 14]) For a set H and $n \in \mathbb{N}_0$, the iterated (or n -th) powerset is defined recursively by

$$\mathcal{P}_0(H) := H, \quad \mathcal{P}_{n+1}(H) := \mathcal{P}(\mathcal{P}_n(H)).$$

We also write $\mathcal{P}^n(H)$ for $\mathcal{P}_n(H)$. The *nonempty* iterated powerset is obtained by removing the empty set at each step:

$$\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}, \quad \mathcal{P}_1^*(H) := \mathcal{P}^*(H), \quad \mathcal{P}_{n+1}^*(H) := \mathcal{P}^*(\mathcal{P}_n^*(H)).$$

Example 1 (Second Powerset and σ -Algebras in a Coin Toss). Let the sample space for a single fair coin toss be

$$\Omega = \{H, T\}.$$

The first powerset,

$$\mathcal{P}_1(\Omega) = \mathcal{P}(\Omega) = \{\emptyset, \{H\}, \{T\}, \{H, T\}\},$$

is the family of all events. The second powerset,

$$\mathcal{P}_2(\Omega) = \mathcal{P}(\mathcal{P}_1(\Omega)),$$

is the set of all *collections of events*; among these, the collections that satisfy the axioms of a σ -algebra are precisely the σ -algebras on Ω . For example,

$$\mathcal{F}_{\text{trivial}} = \{\emptyset, \{H, T\}\} \in \mathcal{P}_2(\Omega), \quad \mathcal{F}_{\text{full}} = \mathcal{P}_1(\Omega) \in \mathcal{P}_2(\Omega).$$

Here $\mathcal{F}_{\text{trivial}}$ and $\mathcal{F}_{\text{full}}$ denote the trivial and full σ -algebras on Ω , respectively. Thus, the second powerset naturally organizes the possible choices of measurable event families in probability theory.

Definition 5 (Hypergraph). [15, 16] A *hypergraph* is a pair $H = (V, E)$ where

- I. V is a nonempty (here: finite) set of *vertices*;
- II. $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a family of nonempty subsets of V , whose elements are called *hyperedges*.

Thus, each hyperedge is a collection of vertices, allowing higher-order (multiway) relations.

Definition 6 (n -SuperHyperGraph). [17, 6] Let V_0 be a finite base set and define the iterated powersets

$$\mathcal{P}^0(V_0) := V_0, \quad \mathcal{P}^{k+1}(V_0) := \mathcal{P}(\mathcal{P}^k(V_0)) \quad (k \geq 0).$$

An n -SuperHyperGraph (with $n \in \mathbb{N}$) is a pair

$$\text{SuHyG}^{(n)} = (V, E),$$

where the n -supervertex set V satisfies $V \subseteq \mathcal{P}^n(V_0)$, and the n -superedge set is a family of nonempty subsets of V :

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Hence each $e \in E$ is a set whose elements are n -supervertices (i.e. members of $V \subseteq \mathcal{P}^n(V_0)$).

Example 2 (2-SuperHyperGraph of Dice Events). Let the sample space for a fair six-sided die be

$$V_0 = \{1, 2, 3, 4, 5, 6\}.$$

Consider three 2-supervertices (elements of $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$):

$$v_1 = \{\{1, 2\}, \{3, 4\}\},$$

$$v_2 = \{\{2, 3\}, \{5, 6\}\},$$

$$v_3 = \{\{1, 2\}, \{5, 6\}\}.$$

Set $V^{(2)} := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$. For each $v \in V^{(2)}$ define its induced event (a subset of V_0)

$$U(v) := \bigcup A.$$

Explicitly,

$$U(v_1) = \{1, 2, 3, 4\}, \quad U(v_2) = \{2, 3, 5, 6\}, \quad U(v_3) = \{1, 2, 5, 6\}.$$

Define the 2-superedges to link supervertices with overlapping induced events:

$$E^{(2)} := \{\{v_1, v_2\}, \{v_2, v_3\}\} \subseteq \mathcal{P}(V^{(2)}) \setminus \{\emptyset\}.$$

Equip each superedge with the probability weight of the overlap under the uniform die:

$$w(\{v_i, v_j\}) := \mathbb{P}(U(v_i) \cap U(v_j)) = \frac{|U(v_i) \cap U(v_j)|}{6}.$$

Therefore

$$\begin{aligned} U(v_1) \cap U(v_2) = \{2, 3\} &\Rightarrow w(\{v_1, v_2\}) = \frac{2}{6} = \frac{1}{3}, \\ U(v_2) \cap U(v_3) = \{2, 5, 6\} &\Rightarrow w(\{v_2, v_3\}) = \frac{3}{6} = \frac{1}{2}. \end{aligned}$$

Thus

$$\text{SuHyG}^{(2)} = (V^{(2)}, E^{(2)})$$

is a 2-SuperHyperGraph with $E^{(2)} \subseteq \mathcal{P}(V^{(2)})$, capturing hierarchical event collections as vertices and their probabilistic overlaps as superedges.

Example 3 (A 3-SuperHyperGraph in Probability: Families of σ -algebras). Let the sample space for two fair coin tosses be

$$V_0 = \Omega = \{HH, HT, TH, TT\}.$$

Level 1 objects are events (subsets of Ω), i.e. elements of $\mathcal{P}^1(\Omega) = \mathcal{P}(\Omega)$.

Level 2 (collections of events). Among all collections in $\mathcal{P}^2(\Omega) = \mathcal{P}(\mathcal{P}(\Omega))$, consider the following σ -algebras:

$$\mathcal{F}_{\text{triv}} = \{\emptyset, \Omega\}, \quad \mathcal{F}_1 = \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\},$$

$$\mathcal{F}_2 = \{\emptyset, \Omega, \{HH, TH\}, \{HT, TT\}\},$$

where $\mathcal{F}_1 = \sigma(\text{first toss})$ and $\mathcal{F}_2 = \sigma(\text{second toss})$.

Level 3 (families of σ -algebras). Form 3-supervertices as *sets of σ -algebras*, i.e. elements of $\mathcal{P}^3(\Omega) = \mathcal{P}(\mathcal{P}^2(\Omega))$:

$$S_A = \{\mathcal{F}_{\text{triv}}, \mathcal{F}_1\}, \quad S_B = \{\mathcal{F}_{\text{triv}}, \mathcal{F}_2\}, \quad S_C = \{\mathcal{F}_1, \mathcal{F}_2\}, \quad S_D = \{\mathcal{F}_{\text{triv}}, \mathcal{F}_1, \mathcal{F}_2\}.$$

Set

$$V := \{S_A, S_B, S_C, S_D\} \subseteq \mathcal{P}^3(\Omega).$$

3-superedges. For each σ -algebra $\mathcal{F} \in \{\mathcal{F}_{\text{triv}}, \mathcal{F}_1, \mathcal{F}_2\}$, let the hyperedge collect all 3-supervertices that *contain* $\mathcal{F}$:

$$e_{\mathcal{F}} := \{S \in V : \mathcal{F} \in S\} \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Explicitly,

$$e_{\text{triv}} = \{S_A, S_B, S_D\}, \quad e_1 = \{S_A, S_C, S_D\}, \quad e_2 = \{S_B, S_C, S_D\}.$$

Define the 3-superedge set

$$E := \{e_{\text{triv}}, e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Conclusion. The pair

$$\text{SuHyG}^{(3)} = (V, E)$$

is a 3-SuperHyperGraph in the sense of the definition: $V \subseteq \mathcal{P}^3(\Omega)$, and each $e \in E$ is a nonempty subset of V . Interpretationally, level 1 objects are events, level 2 objects are σ -algebras (information structures), and level 3 vertices are *families* of such information structures; the superedges group together families sharing a common σ -algebra (e.g. $\sigma(\text{first toss})$).

1.2|HyperNetwork and SuperhyperNetwork

A hypernetwork is a graph generalization where hyperedges connect any number of nodes, enabling modeling of multiway relationships beyond pairwise edges. An n -superhypernetwork uses vertices and hyperedges drawn from the n -th iterated powerset of a base node set to model nested, hierarchical groupings. The definitions of HyperNetwork and SuperhyperNetwork are presented below [18].

Definition 7 (Hypernetwork). [18] A *hypernetwork* is an ordered triple

$$H = (V, \mathcal{E}, w)$$

where

- I. V is a nonempty finite set of *nodes*;
- II. $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is the set of *hyperedges*, each hyperedge $e \in \mathcal{E}$ being a nonempty subset of nodes (allowing multi-node interactions);
- III. $w: \mathcal{E} \rightarrow \mathbb{R}_{\geq 0}$ is a *weight or attribute function* on hyperedges (omitted if unweighted).

A *directed hypernetwork* may be defined by replacing $\mathcal{E} \subseteq \mathcal{P}(V)$ with a set of *ordered* tuples of nodes or by equipping each $e \in \mathcal{E}$ with a head-tail partition. One can further add a *node-labeling* $\ell_V: V \rightarrow L_V$ and a *hyperedge-labeling* $\ell_{\mathcal{E}}: \mathcal{E} \rightarrow L_{\mathcal{E}}$ to record types or properties.

Example 4 (Project Team Hypernetwork (Real-World Collaboration)). Let the node set be employees involved in a product cycle:

$$V = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dan}, \text{Emma}\}.$$

Define hyperedges as project teams (multi-person collaborations), with weights equal to the number of weeks the team worked together:

$$\mathcal{E} = \{e_{\text{dev}} = \{\text{Alice}, \text{Bob}, \text{Carol}\}, e_{\text{ops}} = \{\text{Bob}, \text{Dan}\}, e_{\text{research}} = \{\text{Carol}, \text{Dan}, \text{Emma}\}, e_{\text{launch}} = \{\text{Alice}, \text{Dan}, \text{Emma}\}\}.$$

Let

$$w(e_{\text{dev}}) = 12, \quad w(e_{\text{ops}}) = 5, \quad w(e_{\text{research}}) = 7, \quad w(e_{\text{launch}}) = 3.$$

Then the hypernetwork $H = (V, \mathcal{E}, w)$ captures real team interactions where each hyperedge represents a project team and its duration.

Definition 8 (*n*-SuperHypernetwork). [19, 20, 18] Let V_0 be a finite base set of *nodes*. Define the *n*-th iterated powerset recursively by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)) \quad (k \geq 0).$$

An *n*-superhypernetwork is a tuple

$$\mathcal{N}^{(n)} = (V, \mathcal{E}, w)$$

where

- I. $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n*-supernodes;
- II. $\mathcal{E} \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n*-superedges, each superedge $e \in \mathcal{E}$ being a nonempty subset of V ;
- III. $w: \mathcal{E} \rightarrow \mathbb{R}_{\geq 0}$ is an optional *weight function* assigning a nonnegative real weight (or confidence) to each superedge.

In other words, both vertices and hyperedges of the network are drawn from the *n*-th powerset of the base node set, capturing up to *n* levels of hierarchical grouping.

Example 5 (2-SuperHypernetwork of University Module Clusters). Let the base set of courses be

$$V_0 = \{\text{Math, Physics, CS}\}.$$

We form the second iterated powerset $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$, whose elements are nonempty collections of nonempty course-sets. Choose the following 2-supernodes:

$$\begin{aligned} u_1 &= \{\{\text{Math, Physics}\}\}, \\ u_2 &= \{\{\text{Physics, CS}\}\}, \\ u_3 &= \{\{\text{Math, Physics}\}, \{\text{Physics, CS}\}\}. \end{aligned}$$

Here:

- I. u_1 represents the “Math& Physics” module,
- II. u_2 represents the “Physics& CS” module,
- III. u_3 represents the combined cluster of those two modules.

Define the 2-superhypernetwork

$$\mathcal{N}^{(2)} = (V, \mathcal{E}, w), \quad V = \{u_1, u_2, u_3\}, \quad \mathcal{E} = \{\{u_1, u_2\}\},$$

where the single superedge connects the two module supernodes u_1, u_2 .

Assign a weight function w encoding student enrollment overlap:

$$w(\{u_1, u_2\}) = |\{\text{students enrolled in both modules}\}|.$$

For example, if 30 students take both “Math & Physics” and “Physics & CS,” then $w(\{u_1, u_2\}) = 30$.

Thus $\mathcal{N}^{(2)}$ models hierarchical grouping of courses into modules and clusters, with hyperedges capturing the overlap in student participation across modules.

Example 6 (University Programs as a 3-SuperHypernetwork (Hierarchical Curricula)). Let the base set of *courses* be

$$V_0 = \{\text{CS101, CS102, MATH101, STAT201, ECON101, PSY101}\}.$$

Level 1 (bundles of courses):

$$\begin{aligned} B_1 &= \{\text{CS101, CS102}\}, \quad B_2 = \{\text{MATH101, STAT201}\}, \\ B_3 &= \{\text{ECON101, PSY101}\}, \quad B_4 = \{\text{CS101, STAT201}\}. \end{aligned}$$

Level 2 (programs as sets of bundles):

$$P_\alpha = \{B_1, B_2\}, \quad P_\beta = \{B_2, B_4\}, \quad P_\gamma = \{B_3\}.$$

Level 3 (tracks as sets of programs):

$$S_A = \{P_\alpha, P_\beta\}, \quad S_B = \{P_\beta, P_\gamma\}, \quad S_C = \{P_\alpha, P_\gamma\}.$$

Set the 3-supernode set $V = \{S_A, S_B, S_C\} \subseteq \mathcal{P}^3(V_0)$. Define 3-superedges between tracks that *share at least one program*:

$$\mathcal{E} = \{\{S_A, S_B\}, \{S_A, S_C\}, \{S_B, S_C\}\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Equip each superedge with the weight

$$w(\{S_i, S_j\}) := |S_i \cap S_j| \quad (\text{number of shared programs}).$$

Since

$$S_A \cap S_B = \{P_\beta\}, \quad S_A \cap S_C = \{P_\alpha\}, \quad S_B \cap S_C = \{P_\gamma\},$$

we have

$$w(\{S_A, S_B\}) = w(\{S_A, S_C\}) = w(\{S_B, S_C\}) = 1.$$

Thus the 3-superhypernetwork $\mathcal{N}^{(3)} = (V, \mathcal{E}, w)$ models a real academic setting: courses $\rightarrow$ bundles $\rightarrow$ programs $\rightarrow$ tracks, with superedges encoding shared curricular structure.

1.3|Bayesian Network

A Bayesian network is a directed graph where nodes represent random variables and edges encode conditional dependencies via probability distributions [21, 22, 23, 24].

Definition 9 (Directed Acyclic Graph). (cf.[25, 26]) A *directed acyclic graph* (DAG) is a pair $G = (V, E)$ where $V = \{X_1, \dots, X_n\}$ is a finite set of nodes and $E \subseteq V \times V$ is a set of directed edges $X_i \rightarrow X_j$, such that no sequence of edges forms a cycle. The nodes represent random variables, and an edge $X_i \rightarrow X_j$ indicates that X_i directly influences X_j .

Definition 10 (Bayesian Network). (cf.[27, 28, 29, 30, 31, 32, 33]) A *Bayesian network* over variables $V = \{X_1, \dots, X_n\}$ is a pair $B = \langle G, \Theta \rangle$ where:

- (1) $G = (V, E)$ is a DAG.
- (2) For each i , $\text{Pa}(X_i) \subseteq V \setminus \{X_i\}$ denotes the parents of X_i in G .
- (3) $\Theta = \{\theta_{X_i|\text{Pa}(X_i)}\}$ is a collection of conditional probability distributions. Each

$$\theta_{X_i|\text{Pa}(X_i)} : \text{Dom}(\text{Pa}(X_i)) \longrightarrow \Delta(\text{Dom}(X_i))$$

assigns to each configuration pa_i of the parents a probability mass function over the states of X_i .

Example 7 (Lung Cancer Diagnosis Bayesian Network). Consider the following medical diagnosis scenario with five binary random variables:

$$V = \{\text{Smoker}, \text{Pollution}, \text{Cancer}, \text{XRay}, \text{Dyspnea}\},$$

each taking values in $\{\text{True}, \text{False}\}$. We define a Bayesian network $B = \langle G, \Theta \rangle$ as follows:

Structure G . The DAG over V has edges:

$$\text{Smoker} \rightarrow \text{Cancer}, \quad \text{Pollution} \rightarrow \text{Cancer}, \quad \text{Cancer} \rightarrow \text{XRay}, \quad \text{Cancer} \rightarrow \text{Dyspnea}.$$

Parameters Θ .

I. Prior for Smoker:

$$P(\text{Smoker} = \text{True}) = 0.30, \quad P(\text{Smoker} = \text{False}) = 0.70.$$

II. Prior for Pollution:

$$P(\text{Pollution} = \text{True}) = 0.90, \quad P(\text{Pollution} = \text{False}) = 0.10.$$

III. Conditional for Cancer given its parents:

	Pollution = False	Pollution = True
Smoker = False	0.001	0.03
Smoker = True	0.05	0.10

where each entry is $P(\text{Cancer} = \text{True} \mid \text{Smoker}, \text{Pollution})$.

IV. Conditional for XRay given Cancer:

$$P(\text{XRay} = \text{True} \mid \text{Cancer} = \text{True}) = 0.90, \quad P(\text{XRay} = \text{True} \mid \text{Cancer} = \text{False}) = 0.20.$$

V. Conditional for Dyspnea given Cancer:

$$P(\text{Dyspnea} = \text{True} \mid \text{Cancer} = \text{True}) = 0.65, \quad P(\text{Dyspnea} = \text{True} \mid \text{Cancer} = \text{False}) = 0.30.$$

Joint Factorization. By the Bayesian network structure, the joint distribution factorizes as

$$\begin{aligned} P(\text{Smoker}, \text{Pollution}, \text{Cancer}, \text{XRay}, \text{Dyspnea}) &= P(\text{Smoker}) P(\text{Pollution}) \\ &\quad P(\text{Cancer} \mid \text{Smoker}, \text{Pollution}) P(\text{XRay} \mid \text{Cancer}) \\ &\quad P(\text{Dyspnea} \mid \text{Cancer}). \end{aligned}$$

This network captures how smoking and pollution jointly influence cancer risk, which in turn affects test results and symptom presence.

2|Results of this Paper

In this paper, we present the definitions and key properties of Bayesian-HyperNetworks and Bayesian n -SuperHyperNetworks.

2.1|Bayesian-HyperNetworks

A Bayesian HyperNetwork extends Bayesian networks with directed hyperedges representing conditional distributions over multiple child variables given multiple parent variables.

Definition 11 (Directed Acyclic Hypergraph). (cf.[34, 35]) A *directed acyclic hypergraph* (DAHg) is a pair $H = (V, \mathcal{E})$ where:

- I. $V = \{X_1, \dots, X_n\}$ is a finite set of nodes (random variables).
- II. $\mathcal{E} \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$ is a set of *directed hyperedges*. Each hyperedge $(U \rightarrow W) \in \mathcal{E}$ has a nonempty *tail* $U \subseteq V$ and a nonempty *head* $W \subseteq V$, and no sequence of hyperedges forms a directed cycle.

Example 8 (ETL/ML Data Pipeline as a Directed Acyclic Hypergraph). Consider an analytics workflow where datasets and artifacts are produced in stages. Let each node be a Bernoulli random variable indicating the *availability* of a dataset/artifact:

$$V = \{X_{\text{Raw}}, X_{\text{Users}}, X_{\text{Clean}}, X_{\text{Feat}}, X_{\text{Model}}, X_{\text{Pred}}, X_{\text{Dash}}\}.$$

Interpretations:

X_{Raw} : raw logs available, X_{Users} : user table available,
 X_{Clean} : cleaned events ready, X_{Feat} : feature set built,
 X_{Model} : trained model ready, X_{Pred} : batch predictions ready,
 X_{Dash} : business dashboard refreshed.

Define directed hyperedges $\mathcal{E} \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$ to encode multi-input $\rightarrow$ multi-output production steps:

$$\begin{aligned} e_1 &= (\{X_{\text{Raw}}, X_{\text{Users}}\} \rightarrow \{X_{\text{Clean}}\}), \\ e_2 &= (\{X_{\text{Clean}}\} \rightarrow \{X_{\text{Feat}}\}), \\ e_3 &= (\{X_{\text{Feat}}\} \rightarrow \{X_{\text{Model}}\}), \\ e_4 &= (\{X_{\text{Model}}, X_{\text{Clean}}\} \rightarrow \{X_{\text{Pred}}\}), \\ e_5 &= (\{X_{\text{Pred}}, X_{\text{Users}}\} \rightarrow \{X_{\text{Dash}}\}). \end{aligned}$$

Set $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$.

Acyclicity. Define a level function $L : V \rightarrow \mathbb{N}$ by

$$L(X_{\text{Raw}}) = L(X_{\text{Users}}) = 0, \quad L(X_{\text{Clean}}) = 1, \quad L(X_{\text{Feat}}) = 2, \quad L(X_{\text{Model}}) = 3, \quad L(X_{\text{Pred}}) = 4, \quad L(X_{\text{Dash}}) = 5.$$

For each hyperedge $(U \rightarrow W) \in \mathcal{E}$,

$$\max_{u \in U} L(u) < \min_{w \in W} L(w),$$

so no directed cycle can occur. Hence $H = (V, \mathcal{E})$ is a directed acyclic hypergraph representing a real-world data pipeline: raw inputs and reference tables $\rightarrow$ cleaned data $\rightarrow$ features $\rightarrow$ trained model $\rightarrow$ predictions $\rightarrow$ dashboard.

Definition 12 (Bayesian HyperNetwork). A *Bayesian HyperNetwork* is a pair $\text{BHN} = \langle H, \Theta \rangle$ where:

- I. $H = (V, \mathcal{E})$ is a directed acyclic hypergraph.
- II. $\Theta = \{\theta_{W|U} : \text{Dom}(U) \rightarrow \Delta(\text{Dom}(W))\}_{(U \rightarrow W) \in \mathcal{E}}$ assigns to each hyperedge $(U \rightarrow W)$ a conditional distribution of the head variables W given the tail variables U .

The *global factorization* over all variables in V is

$$P(V) = \prod_{(U \rightarrow W) \in \mathcal{E}} P(W | U) = \prod_{(U \rightarrow W) \in \mathcal{E}} \theta_{W|U}(W, U).$$

Example 9 (Bayesian HyperNetwork for Medical Diagnosis). Consider three binary random variables:

$$V = \{\text{Fever}, \text{Cough}, \text{Disease}\},$$

each taking values in $\{\text{Yes}, \text{No}\}$. We model their joint distribution with a Bayesian HyperNetwork $\text{BHN} = \langle H, \Theta \rangle$ defined as follows:

Hypergraph H .

$$H = (V, \mathcal{E}), \quad \mathcal{E} = \{(\{\text{Fever}\} \rightarrow \{\text{Fever}\}), (\{\text{Cough}\} \rightarrow \{\text{Cough}\}), (\{\text{Fever}, \text{Cough}\} \rightarrow \{\text{Disease}\})\}.$$

Here we use two trivial hyperedges to represent root marginals, and one nontrivial hyperedge $\{\text{Fever}, \text{Cough}\} \rightarrow \{\text{Disease}\}$ for the diagnosis.

Parameter set Θ .

- I. Marginal for Fever: $\theta_{\{\text{Fever}\}|\{\text{Fever}\}}$ with

$$P(\text{Fever} = \text{Yes}) = 0.10, \quad P(\text{Fever} = \text{No}) = 0.90.$$

- II. Marginal for Cough: $\theta_{\{\text{Cough}\}|\{\text{Cough}\}}$ with

$$P(\text{Cough} = \text{Yes}) = 0.15, \quad P(\text{Cough} = \text{No}) = 0.85.$$

- III. Conditional for Disease: $\theta_{\{\text{Disease}\}|\{\text{Fever}, \text{Cough}\}}$ given by

	Cough = No	Cough = Yes
Fever = No	0.01	0.05
Fever = Yes	0.30	0.90

where each entry is $P(\text{Disease} = \text{Yes} | \text{Fever}, \text{Cough})$.

Global factorization. The joint probability factors as

$$P(\text{Fever}, \text{Cough}, \text{Disease}) = P(\text{Fever}) P(\text{Cough}) P(\text{Disease} \mid \text{Fever}, \text{Cough}).$$

This Bayesian HyperNetwork captures both simple marginals (via trivial hyperedges) and the complex dependence of disease on symptoms (via a directed hyperedge from $\{\text{Fever}, \text{Cough}\}$ to $\{\text{Disease}\}$).

Theorem 1. *Every Bayesian network and every (undirected) hypernetwork can be embedded as a special case of a Bayesian HyperNetwork.*

Proof: (a) **Generalizing Bayesian networks.** If we restrict $\mathcal{E}$ to hyperedges $(\{X_i\} \rightarrow \{X_j\})$ of size two and let each $\theta_{\{X_j\}|\{X_i\}}$ coincide with the CPT $P(X_j \mid X_i)$, then BHN reduces exactly to a Bayesian network over the same DAG structure.

(b) **Generalizing hypernetworks.** If we take each $\theta_{W|U}$ to be the trivial distribution assigning probability one to a single state of W (i.e. no uncertainty), then the joint factorization imposes no probabilistic constraints, and BHN collapses to the underlying directed acyclic hypergraph H . Hence every hypernetwork (when oriented and acyclic) is recovered.

2.2|Bayesian n -SuperHyperNetworks

A Bayesian n -SuperHyperNetwork generalizes hypernetworks further by using n -th level superedges to specify conditional distributions among hierarchical groups of variables.

Definition 13 (Bayesian n -SuperHyperNetwork). A *Bayesian n -SuperHyperNetwork* is a pair $\text{BnSHN} = \langle \text{NHN}^{(n)}, \Theta \rangle$ where:

- I. $\text{NHN}^{(n)} = (V, \mathcal{E}, w)$ is an n -SuperHyperNetwork.
- II. $\Theta = \{\theta_{W|U} : \text{Dom}(U) \rightarrow \Delta(\text{Dom}(W))\}_{(U \rightarrow W) \in \mathcal{E}}$ assigns to each n -superedge $(U \rightarrow W)$ a conditional distribution over the n -supernodes W given U .

The joint distribution factorizes as

$$P(V) = \prod P(W \mid U).$$

Example 10 (Bayesian 2-SuperHyperNetwork for Symptom Cluster Diagnosis). Let the base set of binary symptoms be

$$V_0 = \{\text{Fever}, \text{Cough}, \text{Fatigue}\},$$

each taking values in $\{\text{Present}, \text{Absent}\}$. The second powerset $\mathcal{P}^2(V_0)$ consists of nonempty subsets of nonempty subsets of V_0 . We select the following 2-supervertices:

$$V = \{u_1 = \{\{\text{Fever}\}, \{\text{Cough}\}\}, u_2 = \{\{\text{Cough}\}, \{\text{Fatigue}\}\}, u_3 = \{\{\text{Fever}\}, \{\text{Fatigue}\}\}\}.$$

Each u_i represents a “two-symptom cluster” event. Define a single 2-superedge:

$$e = \{u_1, u_2\} \longrightarrow \{u_3\},$$

indicating that the joint presence of clusters $\{\text{Fever}, \text{Cough}\}$ and $\{\text{Cough}, \text{Fatigue}\}$ influences the cluster $\{\text{Fever}, \text{Fatigue}\}$.

We assign a weight function $w(e) = 1$ (ignored for inference) and specify the conditional distribution

$$\theta_{\{u_3\}|\{u_1, u_2\}} : \{\text{Present}, \text{Absent}\}^2 \longrightarrow \Delta(\{\text{Present}, \text{Absent}\})$$

as follows:

$$\begin{aligned} P(u_3 = \text{Present} \mid u_1 = \text{Present}, u_2 = \text{Present}) &= 0.80, \\ P(u_3 = \text{Present} \mid u_1 = \text{Present}, u_2 = \text{Absent}) &= 0.30, \\ P(u_3 = \text{Present} \mid u_1 = \text{Absent}, u_2 = \text{Present}) &= 0.25, \\ P(u_3 = \text{Present} \mid u_1 = \text{Absent}, u_2 = \text{Absent}) &= 0.05, \end{aligned}$$

with the complementary probability for **Absent**. The joint distribution over all 2-supervertices factorizes as

$$P(u_1, u_2, u_3) = P(u_1) P(u_2) P(u_3 | u_1, u_2),$$

where $P(u_1)$ and $P(u_2)$ are prior marginals (e.g. set to 0.2–0.4 based on epidemiological data).

This Bayesian 2-SuperHyperNetwork models how overlapping two-symptom clusters combine to influence the emergence of another cluster, capturing hierarchical interactions in medical diagnosis.

Example 11 (Bayesian 2-SuperHyperNetwork in Supply Chain Reliability). Let the base set of suppliers be

$$V_0 = \{S1, S2, S3\},$$

each with a binary state $\{\text{Up}, \text{Down}\}$. We form the second iterated powerset $\mathcal{P}^2(V_0)$ and choose three 2-supervertices:

$$u_1 = \{\{S1\}, \{S2\}\},$$

$$u_2 = \{\{S2\}, \{S3\}\},$$

$$u_3 = \{u_1, u_2\}.$$

Here u_1 represents the cluster of suppliers S1 and S2, u_2 the cluster of S2 and S3, and u_3 the meta-cluster combining those two clusters.

Define the directed 2-superhypergraph

$$\text{NHN}^{(2)} = (V, \mathcal{E}, w), \quad V = \{u_1, u_2, u_3\}, \quad \mathcal{E} = \{(\{u_1, u_2\} \rightarrow \{u_3\})\},$$

with trivial weight $w \equiv 1$.

Assign the parameter set Θ as follows:

- I. Prior reliability of cluster u_1 : $\theta_{u_1|\emptyset}$: $P(u_1 = \text{Up}) = 0.90$, $P(u_1 = \text{Down}) = 0.10$.
- II. Prior reliability of cluster u_2 : $\theta_{u_2|\emptyset}$: $P(u_2 = \text{Up}) = 0.85$, $P(u_2 = \text{Down}) = 0.15$.
- III. Conditional reliability of meta-cluster u_3 given u_1, u_2 :

$$\theta_{u_3|\{u_1, u_2\}}(u_3, u_1, u_2) = P(u_3 = \text{Up} | u_1, u_2),$$

with

	$u_2 = \text{Down}$	$u_2 = \text{Up}$
$u_1 = \text{Down}$	0.20	0.50
$u_1 = \text{Up}$	0.60	0.95

and complementary probabilities for $u_3 = \text{Down}$.

The joint distribution factorizes as

$$P(u_1, u_2, u_3) = P(u_1) P(u_2) P(u_3 | u_1, u_2),$$

which models how individual cluster reliabilities propagate through the hierarchical network to determine overall system reliability.

Example 12 (Bayesian 2-SuperHyperNetwork for Financial Portfolio Risk). Let the base set of assets be

$$V_0 = \{A, B, C\},$$

each with binary return state $\{\text{High}, \text{Low}\}$. We form the second iterated powerset $\mathcal{P}^2(V_0)$ and select three 2-supervertices:

$$u_1 = \{\{A\}, \{B\}\},$$

$$u_2 = \{\{B\}, \{C\}\},$$

$$u_3 = \{\{A\}, \{C\}\}.$$

Here u_1 represents the cluster of assets A and B, u_2 the cluster of B and C, and u_3 the cluster of A and C.

Define the directed 2-superhypergraph

$$\text{NHN}^{(2)} = (V, \mathcal{E}, w), \quad V = \{u_1, u_2, u_3\}, \quad \mathcal{E} = \{(\{u_1, u_2\} \rightarrow \{u_3\})\},$$

with uniform weight $w \equiv 1$.

Assign the parameter set Θ as follows:

I. Prior marginal for u_1 :

$$P(u_1 = \text{High}) = 0.55, \quad P(u_1 = \text{Low}) = 0.45.$$

II. Prior marginal for u_2 :

$$P(u_2 = \text{High}) = 0.60, \quad P(u_2 = \text{Low}) = 0.40.$$

III. Conditional distribution for u_3 given u_1, u_2 :

$$\theta_{\{u_3\}|\{u_1, u_2\}} = P(u_3 = \text{High} \mid u_1, u_2),$$

specified by

	$u_2 = \text{Low}$	$u_2 = \text{High}$
$u_1 = \text{Low}$	0.20	0.50
$u_1 = \text{High}$	0.65	0.95

with complementary probabilities for $u_3 = \text{Low}$.

The joint probability over all 2-supervertices factorizes as

$$P(u_1, u_2, u_3) = P(u_1) P(u_2) P(u_3 \mid u_1, u_2).$$

This Bayesian 2-SuperHyperNetwork models how the performance clusters of assets A/B and B/C combine to influence the risk cluster of A/C, capturing hierarchical dependencies in portfolio risk assessment.

Theorem 2. *Bayesian n -SuperHyperNetworks simultaneously generalize Bayesian networks, Bayesian HyperNetworks, and n -SuperHyperNetworks.*

Proof: BnSHN reduces to:

- I. A Bayesian HyperNetwork when $n = 1$ (since $\P^1(V_0) = \P(V_0)$).
- II. A Bayesian network when $n = 1$ and all hyperedges have tails and heads of size one.
- III. An n -SuperHyperNetwork when each $\theta_{W|U}$ is trivial (unit mass).

Thus all three structures arise as special cases.

Theorem 3 (Local Markov Property). *Let $\text{BnSHN} = (\text{NHN}^{(n)}, \Theta)$ be a Bayesian n -SuperHyperNetwork on supernodes V and superedges $\mathcal{E}$. For any supernode $W \in V$ with parent supernodes $\text{Pa}(W) = U$, define the set of non-descendant, non-parent supernodes as*

$$\text{ND}(W) = V \setminus (\text{Desc}(W) \cup \text{Pa}(W)).$$

Then

$$W \perp\!\!\!\perp \text{ND}(W) \mid \text{Pa}(W)$$

in the joint distribution $P(V) = \prod P(X \mid U)$.

Proof: By construction, the joint density factorizes as

$$P(V) = \prod_{(U \rightarrow X) \in \mathcal{E}} P(X \mid U).$$

Consider the conditional distribution of W given all other supernodes:

$$P(W \mid V \setminus \{W\}) = \frac{P(V)}{\sum_w P(V)} = P(W \mid \text{Pa}(W)),$$

because only the factor $P(W \mid \text{Pa}(W))$ involves W ; all other factors either do not depend on W or cancel in the normalization. Hence, once $\text{Pa}(W)$ is known, W is independent of every other supernode not in $\text{Desc}(W)$ or $\text{Pa}(W)$, i.e. $W \perp\!\!\!\perp \text{ND}(W) \mid \text{Pa}(W)$.

Theorem 4 (Global Markov Property). *Let $A, B, C \subseteq V$ be disjoint sets of supernodes. Form the moral graph of $\text{NHN}^{(n)}$ by*

- I. *Replacing each directed hyperedge $(U \rightarrow W)$ by undirected edges between every pair in $U \cup W$.*
- II. *“Marrying” all parents in U (making U a clique) and dropping edge directions.*

If every undirected path from any $a \in A$ to any $b \in B$ intersects C , then in the joint distribution,

$$A \perp\!\!\!\perp B \mid C.$$

Proof: The moral graph construction ensures that all conditional dependencies implied by directed hyperedges are represented as undirected connections. A classic separation argument in undirected graphs then applies: if C blocks all paths between A and B , no factor in the factorization can transmit probabilistic influence between them once C is observed. Therefore $P(A, B \mid C) = P(A \mid C) P(B \mid C)$, establishing $A \perp\!\!\!\perp B \mid C$.

Theorem 5 (Parameter Independence under Dirichlet Priors). *Assume each conditional distribution $\theta_{W|U}$ in Θ has a conjugate Dirichlet prior*

$$\theta_{W|U} \sim \text{Dir}(\alpha_{w|u} : w \in \text{Dom}(W)).$$

Given independent data counts $N_{w,u}$ for each configuration $(U = w)$, the posterior factorizes as

$$p(\Theta \mid \{N_{w,u}\}) = \prod \text{Dir}(\alpha_{w|u} + N_{w,u} : w \in \text{Dom}(W)).$$

Proof: Because the joint likelihood factorizes $\prod_{(U \rightarrow W)} \prod_{w,u} \theta_{W|U}(w \mid u)^{N_{w,u}}$, and each prior over $\theta_{W|U}$ is Dirichlet, posterior conjugacy implies

$$p(\theta_{W|U} \mid \{N_{w,u}\}) \propto \left[\prod_{w,u} \theta_{W|U}(w \mid u)^{\alpha_{w|u}-1} \right] \left[\prod_{w,u} \theta_{W|U}(w \mid u)^{N_{w,u}} \right] = \text{Dir}(\alpha_{w|u} + N_{w,u}).$$

Since factors share no parameters, the full posterior factorizes across hyperedges.

Theorem 6 (Consistency under Aggregation). *If two Bayesian n -SuperHyperNetworks BnSHN_1 and BnSHN_2 share the same hypergraph structure but differ in variable granularity—where BnSHN_2 coarsens supernodes of BnSHN_1 —then marginalizing the finer network over the aggregated states recovers the coarser network’s joint distribution.*

Proof: Aggregation corresponds to summing out detailed supernodes in the factorized distribution of BnSHN_1 . Because the factorization aligns with the coarser network’s hyperedges (which are unions of fine-level hyperedges), the resulting marginalized distribution has the same factorization as BnSHN_2 , ensuring consistency.

3|Conclusion

In this work, we introduced the concepts of *Bayesian hypernetworks* and *Bayesian superhypernetworks*, which extend Bayesian networks by incorporating hypernetwork and superhypernetwork structures. In the future, we expect further research on extended concepts of *Bayesian hypernetworks* and *Bayesian superhypernetworks* using Fuzzy Sets[36], Intuitionistic Fuzzy Sets[37], Neutrosophic Sets[38, 39], and Plithogenic Sets[40], as well as the design of graph algorithms, computational experiments with datasets, and the formal definition of operations.

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Data Availability

This manuscript presents purely conceptual work without empirical data. Scholars interested in these ideas are invited to undertake experimental or case-study research to substantiate and extend the proposed frameworks.

Ethical Approval

This paper involves no human or animal subjects and thus did not require ethics committee review or approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors declare that there are no competing interests concerning the content or publication of this article.

Disclaimer

The theoretical models and propositions herein have not yet been subjected to practical validation. Readers should independently verify all citations and be aware that inadvertent inaccuracies may remain. The opinions expressed are those of the authors and do not necessarily represent the views of affiliated organizations.

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